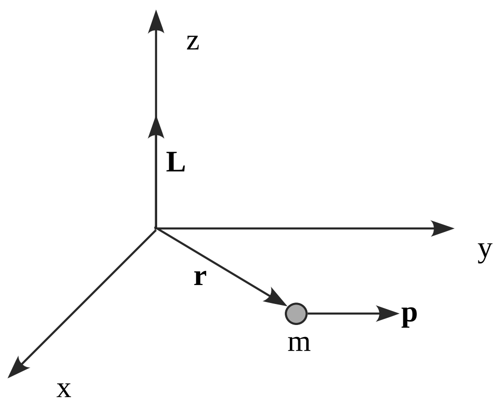
QUANTIZATION OF ANGULAR MOMENTUM

Recall that classically the angular momentum is given by **L** = **r** x **p.**



$$-\left[\frac{1}{f(\theta)\,sin\,\theta}\frac{d}{d\theta}\left(sin\,\theta\frac{df(\theta)}{d\theta}\right)+\frac{1}{g(\varphi)\,sin^2\,\theta}\frac{d^2g(\varphi)}{d\varphi^2}\right]=\ell(\ell+1)$$

multiply both sides by: $\hbar^2 f(\theta)g(\varphi)$

$$-\hbar^2\left[\frac{g(\varphi)}{sin\,\theta}\frac{d}{d\theta}\left(sin\,\theta\frac{df(\theta)}{d\theta}\right)+\frac{f(\theta)}{sin^2\,\theta}\frac{d^2g(\varphi)}{d\varphi^2}\right]=\ell(\ell+1)Y_\ell^m\hbar^2$$

$$-\hbar^2\left[\frac{1}{sin\,\theta}\frac{\partial}{\partial\theta}\left(sin\,\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{sin^2\,\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_\ell^m=\ell(\ell+1)Y_\ell^m\hbar^2$$

We now define the angular momentum operator L^2 by:

$$\left(L^2\right)_{op}=-\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Thus, the above equation can be written as:

$$\left(L^2\right)_{op}Y_\ell^m=\ell(\ell+1)\hbar^2Y_\ell^m$$

From which it follows that,

$$L^2 = \ell(\ell+1)\hbar^2$$

(1) $\boxed{L = \sqrt{\ell(\ell+1)}\hbar \ , \ \ \ell = 0,1,2...,n-1}$ Quantization of Angular Momentum

Likewise, it can also be shown that the z – component of angular momentum is also quantized and given by:

(2) $\boxed{L_z = m\hbar \ , \ m = 0,\pm 1,\pm 2,....,\pm \ell}$

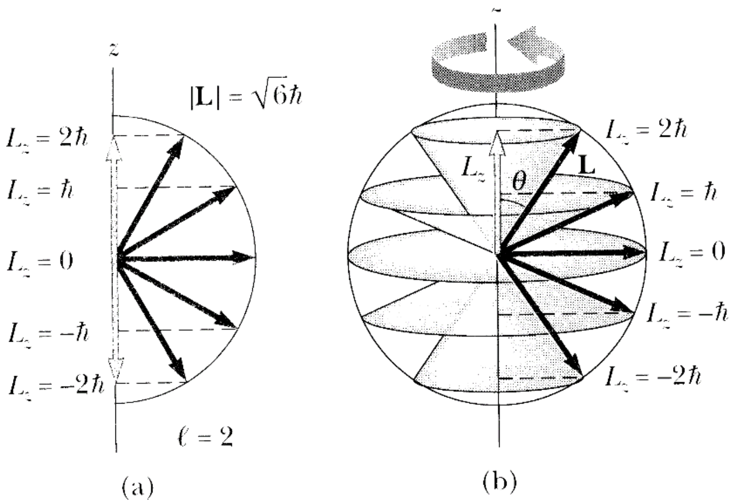
Thus, for all potential energies where U=U(r) the angular momentum will be quantized and given by the above equations.

SPACE QUANTIZATION

The physical significance of equations (1) and (2) above is that the angular momentum vector **L** can only point in those directions *in space* such that the projection of **L** onto the z-axis is one of the values given by equation (2). Thus, we say that L is **space-quantized**.

Ex. Consider the case for which ℓ = 2.

ℓ	m_ℓ	$\lvert\vec{L}\rvert$	L_z
2	-2	$\sqrt{6}\hbar$	-2$\hbar$
2	-1	$\sqrt{6}\hbar$	-$\hbar$
2	0	$\sqrt{6}\hbar$	0
2	1	$\sqrt{6}\hbar$	$\hbar$
2	-2	$\sqrt{6}\hbar$	2$\hbar$



Note that the angular momentum vector **L** never points in the z-direction since L_z must be smaller than the magnitude of **L**. This is a consequence of the uncertainty principle for angular momentum that implies that no two components of **L** can be known precisely.

From a 3-D perspective **L** processes around the z-axis so as to trace out a cone at angle θ in space.

$$\boxed{\cos\theta = \frac{L_z}{L} = \frac{m_\ell}{\sqrt{\ell(\ell+1)}}}$$ Space Quantization of **L**